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INTEGRAL CLOSURE OF UNSTABLE STEENROD ALGEBRA ACTIONS

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Dedicated to George Cooke, friend and tutor

Integral domains with an action of the Steenrod algebra $\mathscr{A}_p$ frequently arise in the study of the cohomology of classifying spaces of Lie groups and associated spaces. For the most part algebraic topologists have considered only the case of polynomial algebras and results in this case have been more numerical than conceptual. The first work to exploit the interaction of the algebro-geometric properties of the underlying cohomology algebra with the Steenrod algebra action was provided for products of infinite complex projective spaces by Serre [8] (assume that p is odd for this paper):

Theorem A. *Let $\mathscr{P}$ be an homogeneous prime ideal of $H^*((\mathbb{C}P^\infty)^n, \mathbb{F}_p)$ such that $\mathscr{P}$ is invariant ($\mathscr{A}_p\mathscr{P} \subset \mathscr{P}$). Then $\mathscr{P}$ is generated by elements of grading two.*

Later, Quillen in his study of the equivariant cohomology ring [7] demonstrated a similar connection between the $\mathscr{A}_p$-invariant prime ideals of $H^*(BH, \mathbb{F}_p)$ and the elementary p-subgroups of H, for H a not necessarily connected Lie group H. Then the results of Adams–Mahmud [2] on maps between classifying spaces provided encouraging evidence that globally accessible algebraic information was obtainable from the unstable $\mathscr{A}_p$-actions. Motivated by this evidence and the desire to find a conceptual approach to the classification problem for polynomial algebras with an unstable action of the Steenrod algebra, the author began in [9] a general study of $\mathscr{A}_p$-actions as a special case of differential algebra in characteristic p. One dividend of the study was a short proof of some of the Adams-Mahmud results using Theorem A, [9].

One lesson of the study was that extensions of standard algebraic constructions to the setting of integral domains with $\mathscr{A}_p$-actions could be profitable. There, the

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processes of localization and separable extensions were applied. In the present work we focus on properties of an integral domain B in its field of fractions K, for the case that B has an unstable $\mathscr{A}_p$-action. Using the Cartan formula one can show that the $\mathscr{A}_p$-action on B extends to an action on K (which is *not* unstable). In K, there may be two special classes of elements:

(a) those which satisfy a monic polynomial with coefficients in B (elements *integral* over B) and

(b) those which are finite dimensional from the viewpoint of the Steenrod algebra action.

We establish that the unstable $\mathscr{A}_p$-action on B always extends to an unstable $\mathscr{A}_p$-action on its graded integral closure $\bar{B}$ in K, and that frequently, $\bar{B}$ contains all the finite dimensional elements of K.

The application of these abstractly satisfying results is to the proof of a strong generalization of Theorem A. Recall from [9] that an equivalent formulation of Theorem A asserts that $H^*((\mathbb{C}P^\infty)^n, \mathbb{F}_p)$ is "algebraically closed" for extensions by two-dimensional elements (again, p odd):

Theorem B [8, 9]. (a) *If B is a finitely generated graded integral domain with an unstable action of $\mathscr{A}_p$ such that B is generated as an algebra over $\mathbb{F}_p$ by the elements of grading two, B_2, then B is isomorphic as algebras with $\mathscr{A}_p$-action to the polynomial $H^*((\mathbb{C}P^\infty)^n, \mathbb{F}_p)$ for $n = \dim_{\mathbb{F}_p} B_2$.*

(b) *If $\varphi : H^*((\mathbb{C}P^\infty)^n, \mathbb{F}_p) \rightarrow B$ is a monic integral extension of graded integral domains with unstable actions of $\mathscr{A}_p$ and B is obtained by adjoining two-dimensional elements to $H^*((\mathbb{C}P^\infty)^n, \mathbb{F}_p)$, then φ is an isomorphism of algebras with $\mathscr{A}_p$-actions.*

The main result is that $H^*((\mathbb{C}P^\infty)^n, \mathbb{F}_p)$ is algebraically closed in the category of graded integral domains with unstable actions of the Steenrod algebra $\mathscr{A}_p$, p odd.

Theorem C. *If $\varphi : H^*((\mathbb{C}P^\infty)^n, \mathbb{F}_p) \rightarrow B$, is a monic integral extension of graded integral domains with unstable $\mathscr{A}_p$-actions, then φ is an isomorphism.*

We will refer to Theorem C as the generalized Serre Lemma. Theorem C is the easier half of a "splitting theorem" conjectured by the author in [9]. A more general form of that conjecture has recently been proved by J.F. Adams [1].

The Embedding Theorem. *If B is a graded integral domain with unstable action of $\mathscr{A}_p$, p-odd, of finite transcendence degree n over $\mathbb{F}_p$, then there exists an algebraic extension $\varphi : B \rightarrow H^*((\mathbb{C}P^\infty)^n, \mathbb{F}_p)$ of algebras with $\mathscr{A}_p$-action.*

The embedding theorem implies theorem C, but the proof presented here is independent of the original proof of the embedding theorem. Theorem C arose from an attempt to understand the implications of the Embedding Theorem. A proof of the Embedding Theorem and applications to the classification problem for polynomial algebras with unstable $\mathscr{A}_p$-actions will appear in joint work with J.F. Adams [3].

Analogues of Theorem C for other cohomology theories seem quite plausible, given the special role of $\mathbb{C}P^\infty$ in these theories. One goal of the present line of proof is to provide an outline which might generalize. Partial analogues for the case of p-adic complex K-theory have been obtained by J.F. Adams [4], and the author.

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1. Recollections

The prime p will be assumed to be odd. With minor modification of indices, the proofs are valid for $p = 2$, and $\mathbb{R}P^\infty$. If p is odd and B is a graded integral domain over $\mathbb{F}_p$, then $B_{2n+1} = 0$, and hence the Bockstein element β of $\mathscr{A}_p$ acts trivially on B. We denote as $\mathscr{B}_p$ the quotient of $\mathscr{A}_p$ by the two-sided ideal generated by the Bockstein. Then $\mathscr{B}_p$ is generated as an algebra by the Steenrod operations P^n of degree $2n(p-1)$, subject to the Adem relations [5]. By an action of $\mathscr{B}_p$ on B, we mean a $\mathbb{F}_p$-vector space map $\rho : \mathscr{B}_p \otimes B \to B$ satisfying the Cartan formula

$$P^n(xy) = \sum P^{n-i}(x)P^i(y).$$

If B is graded and ρ is degree preserving, ρ is said to be an *unstable action* if $P^n x_{2n} = x^p$ and $P^{n+i}x_{2n} = 0$ for $n \geqslant 0$ and $i > 0$. For any action, the total Steenrod class $P_\xi : B \to B[[\xi]]$ defined by $P_\xi x = \sum P^n x \xi^n$ is an injective ring homomorphism. The following properties of integral domains with $\mathscr{B}_p$-action were established in [9] using this property of the total Steenrod class. These do not require the unstable hypothesis for the action of $\mathscr{B}_p$.

Proposition 1.1. (a) *If B is an integral domain with $\mathscr{B}_p$-action, there is a unique extension of the action to the field of fractions of B, K, by $P_\xi(x/y) = P_\xi(x)/P_\xi(y)$.*

(b) *If K is a field with $\mathscr{B}_p$-action, there is an unique extension of the action to each separable algebraic extension field of K.*

(c) *If L is a separable algebraic extension field of K, each automorphism of L over K commutes with the $\mathscr{B}_p$-action. In particular, if θ in L has the property that $P_\xi(\theta) = \theta + \cdots + \theta^p\xi^n$, then each root of the minimal polynomial for θ over K also has this property.*

Proposition 1.1 is a special case of known results for differential fields, e.g. [6].

2. The subalgebra of finite dimensional elements

If B is a graded integral domain with an unstable action of $\mathscr{B}_p$, the action extends to the field of fractions K, by 2.1.a. The action on the field of fractions has no obvious unstable properties. $P^i(x/y)$ will in general be nonzero for infinitely many values of i. However, in contrast to the general action of $\mathscr{B}_p$ on a field, the value of P_ξ is a rational function in ξ by 1.1.a, and for elements of B, a polynomial in ξ. In Section 4, an example of an element of K which is not in B, but for which P_ξ is a polynomial in ξ will be given. This is a useful concept to formalize here.

Definition 2.1. Let K be a field with $\mathscr{B}_p$-action. Define $K_{(0)} = \mathbb{F}_p$ and $K_{(2n)}$ to be the set of all x in K such that $P^n x = x^p$ and $P^{n+i}x = 0$ for all $i > 0$. $K^f = \bigoplus K_{(2n)}$ is then a graded integral domain.

If one modifies the definition of a $2n$-dimensional element to include the condition that it be annihilated by all Steenrod operations of excess greater than $2n$, one could assert that K^f has an unstable action of $\mathscr{B}_p$. This is not required for the sequel.

Proposition 2.2. *If $K \to L$ is an algebraic extension of fields with $\mathscr{B}_p$-action, then $K^f \to L^f$ is an integral extension.*

Proof. (a) Suppose that $K \to L$ is separable. If θ in $L_{(2n)}$ has minimal polynomial f over K, $\sum a_i X^{N-i}$, with $a_0 = 1$, we must show that a_i belongs to K^f. By the results of Section 1, we may assume that L is a splitting field for $f(X)$. Each root of $f(X)$ is $2n$-dimensional in L. But a_i is the i-th elementary symmetric polynomial in the roots of $f(X)$, and hence must be finite dimensional.

(b) Suppose that $K \to L$ is purely inseparable. Then for θ in L^f, there exists $j \geq 0$ such that $\theta^{p^j} = x$, for some x in K. Since $P_\xi x = (P_\xi \theta)^{p^j}$, x is in K^f.

(c) The extension $K \to L$ can be factored as $K \to L'$ a separable extension, and $L' \to L$ a purely inseparable extension. By Section 1, L' inherits an unique $\mathscr{B}_p$-action, so parts a) and b) apply.

Proposition 2.3. *If K is the field of fractions of a graded integral domain B with an unstable action of $\mathscr{B}_p$, then $K_{(2n)}$ is contained in the homogeneous portion*

$$(S^{-1}B)_{2n} = \{x_{2n+i}/y_i \text{ for } x \text{ in } B_{2n+i},\ y \text{ in } B_i\}.$$

If $z \in (S^{-1}B)_{2n}$ and $P_\xi(z)$ is a polynomial, $z \in K_{(2n)}$.

Proof. Suppose that R/S is in $K_{(2n)}$, and that $R = x_{2N}$ + lower degree terms, and $S = y_{2M}$ + lower degree terms. Then $P_\xi(R/S)P_\xi(S) = P_\xi(R)$ in $K[\xi]$, the polynomials in ξ with K coefficients. Equating the highest powers of ξ, we see that $y^p(R/S)^p = x^p$. That is, $M + n = N$, and $R/S = x/y$. Similarly $P_\xi(z)P_\xi(y) = P_\xi(X)$ implies $P^n z = z^p$ and $P^{n+i}z = 0$, for $z = x/y \in (S^{-1}B)_{2n}$.

3. Integral closure for unstable actions

The fact that there may be finite dimensional elements in K that are not in B may seem surprising, but examples are easily constructed: Let B be the subalgebra of $H^*(\mathbb{C}P^\infty, \mathbb{F}_p) = \mathbb{F}_p[T]$ consisting of the elements of dimension 4 or larger. Thus T is not in B, so T^3/T^2 is an element of K which is not in B. However T satisfies an equation of integral dependence $X^2 - T^2 = 0$ over B. We will establish a connection between the finite dimensional elements of K and the integral closure of B in K.

Definition 3.1. Let B be a graded integral domain with an unstable action of $\mathcal{B}_p$. Define $\bar{B}$ as the intersection of $S^{-1}B$ with the integral closure of B in K, the field of fractions of B.

Proposition 3.2. *$\bar{B}$ is a graded integral domain with an unstable action of $\mathcal{B}_p$.*

Proof. We show that $\bar{B}$ is contained in K^f: let $(x/y)^N + \cdots + a_N = 0$ be an equation of integral dependence over B. Applying P_ξ, we nave an equation of integral dependence of $P_\xi(x/y)$ over $K[\xi]$ (in fact, over $B[\xi]$). But $K[\xi]$ is integrally closed, so $P_\xi(x/y)$ is a polynomial in ξ. Hence, from the relation $P_\xi(y)P_\xi(x/y) = P_\xi(x)$, it follows that x/y is in K^f. On the other hand, the coefficients of $P_\xi(x/y)$, $P^i(x/y)$ are integral over B, since $P_\xi(x/y)$ is integral over $B[\xi]$. That is, $P^i\bar{B}$ is contained in $\bar{B}$. Thus $\bar{B}$ has an unstable action of $\mathcal{B}_p$.

Proposition 3.3. *If B is a graded unique factorization domain with an unstable action of $\mathcal{B}_p$, $K^f = B$, for K the field of fractions of B.*

Proof. If x/y is in $K_{(2n)}$, we can assume that x is in B_{2n+i} and y is in B_i, by 2.3. It suffices to consider the case that y is prime in B. For a polynomial $f(\xi) = \sum a_i\xi^i$, $K[\xi]$, the y-content is defined as y^s, for $s = \min_i(\nu_y(a_i))$, for $\nu_y(-)$ the valuation at y. For $P_\xi(y)$, the y-content is y^0 or y^1. The y-content of $P_\xi(x/y) = x/y + \cdots + (x/y)^p\xi^n$ is y^{-q} for some $q \geqslant p$, if y does not divide x in B. By Gauss's Lemma, $P_\xi(y)P_\xi(x/y) = P_\xi(x)$ implies that y-content $P_\xi(x) = y^{-q+0}$ or y^{-q+1}. But $P_\xi(x)$ is in $B[\xi]$, so this is impossible. That is, y must divide x in B.

One suspects that 3.3 is true for B merely integrally closed. Corollary 3.4 confirms this for the case needed in Section 4.

Corollary 3.4. *If C is a graded integral domain with an unstable action of $\mathcal{B}_p$ which is a graded integral extension of an* UFD *B with an unstable action of $\mathcal{B}_p$, as algebras with $\mathcal{B}_p$-actions, then $\bar{C} = L^f$, where L is the field of fractions of C.*

Proof. By 3.2, $\bar{C}$ is contained in L^f. By 2.2, L^f is integral over K^f. But $B = K^f$ by 3.3, so L^f is integral over B, and hence over C. Thus L^f is contained in $\bar{C}$, and $\bar{C} = L^f$.

4. The generalized Serre lemma

If B is an integral extension of $H^*((\mathbb{C}P^\infty)^n, \mathbb{F}_p) = \mathbb{F}_p[t_1, \cdots, t_n]$, a natural tactic is seek an induction argument by projecting onto $\mathbb{F}_p[t_2, \ldots, t_n]$. In 4.2, we will see that this requires that Bt_1 be a prime ideal in B.

Proposition 4.1. *If B is a graded integral domain with an unstable action of $\mathcal{B}_p$ such that $B = K^f$, then each two-dimensional element t of B generates a prime ideal of B.*

Proof. Let $x_{2n}y_{2m} = tz$. Since $P_\xi(t) = t + t^p\xi$ is a prime polynomial in $K[\xi]$, it must divide either $P_\xi(x)$, or $P_\xi(y)$. Suppose it divides $P_\xi(x)$. Then x/t is in $K^f = B$. Hence $x = (x/t)t$ is in Bt, and therefore Bt is a prime ideal.

Proposition 4.2. *If $H^*((\mathbb{C}P^\infty)^n, \mathbb{F}_p) = \mathbb{F}_p[t_1, \ldots, t_n] \xrightarrow{\varphi} B$ is an integral extension of graded integral domains with unstable actions of $\mathcal{B}_p$ respecting the $\mathcal{B}_p$-actions, then φ is an isomorphism.*

Proof. The proof will proceed by induction on n.

Case $n = 1$. Let θ in B_{2m} satisfy the minimal polynomial

$$f(\theta) = \theta^N + a_1\theta^{N-1} + \cdots + a_N.$$

Then f is homogeneous, and $a_i = b_i t^{mi}$ in $\mathbb{F}_p[t]$. Set $\lambda = \theta/t^m$. Then λ satisfies the equation $\lambda^N + b_1\lambda^{N-1} + \cdots + b_N = 0$, so λ is in $\bar{\mathbb{F}}_p$. The operations P^i are $\bar{\mathbb{F}}_p$ linear, by 1.1.b, so

$$P^m(\lambda t^m) = \lambda P^m t^m = \lambda t^{mp} = P^m\theta = \theta^p = \lambda^p t^{pm}.$$

That is, $\lambda^p = \lambda$, so λ is in $\mathbb{F}_p$. Hence θ is in $\mathbb{F}_p[t]$.

General case. We may assume that $B = \bar{B}$, by 3.2. By 3.4, we may assume that $B = \bar{B} = K^f$. Hence by 4.1, Bt_1 is a prime ideal of B. But Bt_1 is closed under the action of $\mathcal{B}_p$. Hence the quotients $\mathbb{F}_p[t_1, \ldots, t_n]/(t_1) \to B/(t_1)$ satisfy the hypothesis for $n - 1$, so the extension on quotient rings is trivial. Let θ in B_{2m}, be selected so that θ is not in the image of $\mathbb{F}_p[t_1, \ldots, t_n]$, and m is minimal for this property. θ projects to $g(t_2, \ldots, t_n)$, so $\theta = xt_1 + g$, for some x in B_{2m-2}. But by hypothesis, x is in $\mathbb{F}_p[t_1, \ldots, t_n]$, so θ is also. Thus the extension is trivial.

Note added in proof

Proposition 4.2 is also valid for algebraic extensions, since Corollary 3.4 and Proposition 2.2 imply that an algebraic extension is integral. Also a proof similar to the one of Proposition 3.3 shows that $\bar{B} = K^f$ if B is of finite type. This is not true for B not of finite type, e.g. $B = t_1\mathbb{F}_p[t_1, t_2] \subset \mathbb{F}_p[t_1, t_2]$ (Adams [1]).

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